Amplification and distortion of a periodic rectangular driving signal by a noisy bistable system

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The response of a symmetric bistable system driven by a time periodic rectangular input signal and subject to a white noise is studied. The analysis shows that the stochastic resonant enhancement of a weak amplitude signal implies a distortion of the input shape for intermediate frequencies, due to the dispersivity of the response. On the other hand, the shape can be maintained and the amplitude greatly amplified for very low input frequencies and some values of the noise strengths. These results are corroborated by numerical solutions of the Fokker-Planck equation.

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I. INTRODUCTION

The analysis of the response of stochastic systems driven by time dependent forces has been the subject of numerous studies, both at the theoretical and experimental levels [1]. A large amount of work has been devoted to understanding the phenomenon of stochastic resonance (SR), i.e., the amplification of a weak low frequency sinusoidal signal by the concerted actions of the noise, and the nonlinearity of the system. This phenomenon exposes a qualitative aspect of the noise, which is usually blurred by its diffusive effect. Namely, noise can be looked upon as something useful, in the sense that, because of its presence, it allows a weak input signal to be amplified. These two aspects of the noise have been discussed recently by Dykman *et al.* [2].

In this work we intend to explore further the effects of noise, nonlinearity, and time periodic external forces. We will consider the simple case of an overdamped bistable model subject to an external driving force, which is time periodic but not necessarily sinusoidal. In particular, we will consider the driving field to be a rectangular signal in order to analyze the response of the nonlinear system subject to a complex input with many harmonics. A question addressed in Ref. [2] is the degree of distortion of the shape of the driving field as it becomes amplified by the noise system. We will see that a rectangular signal becomes somewhat distorted for the range of parameters where the stochastic resonance enhancement takes place if the fundamental frequency of the driver is not very low. The distortion depends essentially on the different degrees of amplification of the harmonics of the input and not on the generation of new harmonics. On the other hand, for driving fields with very small fundamental frequencies, we will see that it is possible to get a large amplification of the signal while keeping its shape almost unchanged. This will happen when the noise strength is such that the rate of switching events induced by it is larger than the frequency of the rectangular signal. Another feature to be explored is the possibility of using the enhancement of weak signals at selected frequencies to filter out, to some extent, the other frequency components.

The plan of this paper is as follows. In Sec. II, we briefly analyze the dynamics of the system in the absence of noise. In Sec. III, the stochastic model is presented and the general ideas of the linear response theory (LRT) description of SR are indicated. Situations for which SR cannot adequately be described by LRT will also be discussed. The Fokker-Planck equation (FPE) will be numerically solved and the results discussed in Sec. IV.

II. RESPONSE IN THE ABSENCE OF NOISE

In order to show the noise effects on the dynamics, it is worthwhile to explore the system response when noise is not present. We consider x to be the relevant degree of freedom satisfying the nonlinear evolution equation (in dimensionless form)

$$\frac{dx}{dt} = -U'(x) + f(t) , \tag{1}$$

where the prime refers to the derivative of the bistable potential $U(x) = -x^2/2 + x^4/4$, and where $f(t)$ represents a rectangular signal with period $T = 2\pi/\Omega$ and amplitude S. For driving amplitudes much smaller than the barrier height, the system trajectories are confined within the initial well. Linearization of Eq. (1) around the local minimum shows that the system describes small amplitude oscillations with frequency Ω around the bottom of the well. For large driving amplitudes, linearization is not correct and we analyze the dynamics by numerically solving Eq. (1) by means of a fourth order Runge-Kutta integrator. For each value of Ω, there exists a corresponding amplitude $S^*(\Omega)$ such that for $S < S^*$ the dynamics are still essentially confined within the initial well. On the other hand, for $S > S^*$, the system trajectories explore both wells, describing large amplitude oscillations around the origin. $S^*(\Omega)$ increases with the frequency in such a way that for large frequencies, it is substantially larger than its zero frequency value $S^*(0) = (1/3)^{1/2}$.

Away from the critical line $S^*(\Omega)$, the shape of the response coincides basically with that of the driving field, while near the critical line the output is very distorted. It is still periodic with period T, but the shape of the trajec-

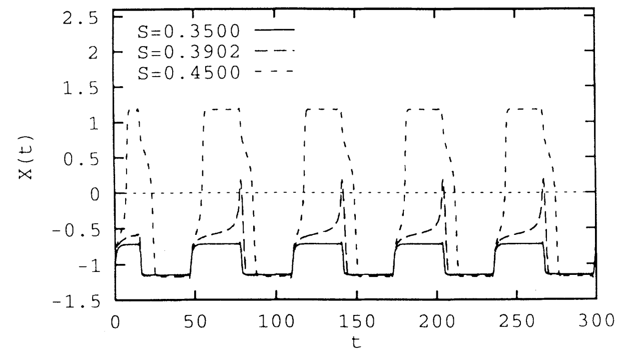

FIG. 1. Deterministic trajectories for different amplitudes of the driving field with $\Omega=0.1$. The solid line corresponds to S smaller than the critical value S^*. The dashed lines correspond to $S \approx S^*$ and $S > S^*$.

tory no longer resembles a rectangular signal, as can be seen in Fig. 1. The rectangular input is generated by a Fourier series with 20 odd harmonics. When S differs from S^*, the response contains the same harmonics as the driving field but with amplitudes that are different from those of the driver. For a very narrow range of S near the critical value, the distortion of the signal is due not only to the different values of the amplitudes of the odd harmonics, but also to the appearance of even ones.

III. THE STOCHASTIC SYSTEM

Let us now include the effect of noise. The system dynamics will be described by the Langevin equation

$$\frac{dx}{dt} = -U'(x) + f(t) + \eta(t) , \tag{2}$$

where $\eta(t)$ is a Gaussian while noise with zero mean and $\langle \eta(t)\eta(s) \rangle = D\delta(t-s)$. The corresponding Fokker-Planck equation for the probability distribution $P(x,t)$ reads

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}[U'(x) - f(t)]P + \frac{D}{2}\frac{\partial^2 P}{\partial x^2} . \tag{3}$$

By contrast with Eqs. (1) and (2), which are nonlinear in the unknown, the FPE is linear. Thus, the Floquet theorem for linear time periodic problems allows us to write $P(x,t)$ in terms of Floquet eigenfunctions and eigenvalues as

$$P(x,t) = \sum_n c_n e^{-\mu_n t} \Phi_n(x,t) , \tag{4}$$

where $\Phi_n(x,t) = \Phi_n(x,t+T)$. The H theorem indicates that the long time solution of the FPE $P_\infty(x,t)$ has a single functional form regardless of the initial preparation of the system. Thus, it has to be periodic in time, i.e.,

$$P_\infty(x,t) = P_\infty(x,t+T) . \tag{5}$$

Consequently, for any values of the noise strength and the parameters of the driving field, the long time average response of the system can be expanded in Fourier form as

$$\langle x(t) \rangle_\infty = \sum_n a_n \cos(n\Omega t + \phi_n) . \tag{6}$$

The shape of the periodic function $\langle x(t) \rangle_\infty$ can in general differ from that of $f(t)$ for two reasons: either because expression (6) contains more harmonics than the ones making up $f(t)$, or because, even if they are the same, each of them is amplified by a different amount.

Let us first consider an external force $f(t)$ with amplitude and period such that its effect on the probability distribution can be described as a small perturbation. Then, the first order perturbation theory of Eq. (3) leads to the LRT result [3]

$$\langle x(t) \rangle_\infty = \int_{-\infty}^{\infty} d\tau K(t-\tau) f(\tau) , \tag{7}$$

with $K(t)=0$ for negative arguments. The response function $K(t)$ depends on the form of the potential $U(x)$ and the noise strength, but it is independent of the driving field parameters. For a rectangular driver with Fourier expansion

$$f(t) = \sum_m f_m \cos(m\Omega t) , \tag{8}$$

one can write

$$\langle x(t) \rangle_\infty = \sum_m \alpha'(m\Omega) f_m \cos(m\Omega t) + \alpha''(m\Omega) f_m \sin(m\Omega t) , \tag{9}$$

where the susceptibility $\alpha(\omega,D) = \alpha'(\omega,D) + \alpha''(\omega,D)$ is the Fourier transform of $K(t)$. Therefore, when LRT applies, Eqs. (6) and (9) show that the harmonics present in the long time response of the system are the same as those in the input signal. As pointed out by Dykman *et al.* [2], we see that linearization does not imply a constant ratio between the response and the driving field. Indeed, LRT yields

$$\begin{aligned} a_m \cos\phi_m &= f_m \alpha'(m\Omega) , \\ -a_m \sin\phi_m &= f_m \alpha''(m\Omega) . \end{aligned} \tag{10}$$

Using the fluctuation-dissipation theorem, the susceptibility can be obtained in terms of the Fourier transform of the equilibrium time correlation function of the unperturbed system. Thus, if this last quantity is available, the long time response of the system can be calculated.

Analytical and numerical works [3,4] indicate that for fixed frequencies smaller than the relaxation frequency within each unperturbed well, the susceptibility has a nonmonotonic behavior with D. It reaches a maximum for a value of the noise strength such that twice the rate of exchange of population between wells induced by this noise (Kramers rate) matches the driving frequency. Thus, the degree of amplification of each driving frequency would be different for a given noise. Harmonics with larger frequencies suffer very little amplification. As a result, LRT predicts that the shape of a rectangular input will not, in general, be kept unchanged due to the different degrees of amplification of the several harmonics. This dispersive character of Eq. (10) opens up the possibility of using a noisy bistable system to separate different frequencies in a weak input signal by adjusting the noise so that other frequency components are filtered out.

Perturbation theory leading to LRT might fail for several reasons. The driving amplitude might get so large that the potential loses its bistable character. Such a situation was analyzed by us in Ref. [5] for a sinusoidal driving field. While a resonant amplification still exists, its effect is less dramatic than for a weak amplitude signal. The enhancement of the amplitude is still due to the matching of the driving frequency with that of the switching events. The transition rate depends on the amplitude of the driving field but, by contrast with the weak amplitude input, the behavior of $\langle x(t)\rangle_\infty$ is not well described by a LRT with a response function constructed from the spectral density of a zero field bistable system.

Perturbation theory also requires the noise strength to be larger than the amplitude of the external field. Thus, if S is small, but $S > D$, the effect of the external field cannot be taken as a small perturbation of the unperturbed dynamics and LRT becomes invalid when $D < S$. Dykman *et al.* [6] have also analyzed the rate equation for the population of the wells in the limit of very small driving frequencies. They find that, in this limit, a sinusoidal input might get amplified and its shape greatly distorted so the output looks like a rectangular signal.

It is interesting to analyze the response of the system to a rectangular input of weak amplitude and very small frequency as one might find strong deviations of the long time probability distribution with respect to the unperturbed equilibrium one. Qualitatively, this can be understood in terms of time scales. If the noise strength D is such that the fundamental external frequency is much smaller than the Kramers rate, the bistable potential remains asymmetric for such a long time that the probability distribution adjusts to a single maximum local equilibrium form around one minimum before the population has a chance to escape the well. When the rectangular input changes sign, there is a noise induced exchange of population, which is very fast on the time scale of the period T, followed by a readjustment of the distribution to the new local equilibrium. Then, the distribution function has a single maximum for large time intervals during which the average value remains large and almost constant. Only during the short time intervals, at which there is exchange of population, will the average change. Thus, the expected response of the system to the weak amplitude rectangular signal should be a large amplitude output with a shape that is similar to the input. As D increases, the amplitude of the response decreases as the diffusive effect of the noise renders the distribution function broader in x space. On the other hand, if D gets so small that the corresponding Kramers rate gets smaller than Ω, the system spends most of its time exchanging population between wells and the distribution function cannot relax to a local equilibrium form so that $P_\infty(x,t)$ will be double humped at all times. Then, one should expect a decrease of the response amplitude for very small D. Clearly, there should be an interval of values for D where the amplification is largest.

When the conditions required for LRT to be valid are not met, even though the long time behavior of the system is periodic in time with period T, there is no guarantee that the response would contain the same harmonics as the driving field. Also, the relation between amplitudes will no longer be given by Eq. (10).

IV. NUMERICAL RESULTS

In order to analyze quantitatively the different regimes indicated above, we have numerically solved the FPE using the split operator method [7] to get the probability distribution and from it, the first few moments. Let us first consider the response of the system to a weak rectangular signal with a small frequency. In Fig. 2, we plot the ratio R of the amplitude of the average response over the amplitude of the driver for $\Omega=0.001$ and $S=0.1$. The phenomenon of SR manifests itself with the maximum of R for a noise strength $D=0.074$ for which twice the Kramers rate $[2\omega^{\mathrm{Kr}}=(\sqrt{2}/\pi)\exp(-1/2D)]$ matches $\Omega/2$. The degree of amplification gets quite large. As discussed before, the qualitative features of the response depend upon whether D is large or small. Next we show results for those different regions.

In Fig. 3, we plot the time evolution of the first two cumulants for $D=0.1$. Very quickly, the average response relaxes to an almost rectangular signal with the same period as the driver but with a much larger amplitude. The second cumulant also shows a periodic behavior with frequency 2Ω, which reflects the symmetry $P(x,t)=P(-x,t+T/2)$ of the distribution function. When $\langle x(t)\rangle \approx \pm 1$, the distribution function has a single peak around the average and $\langle\langle x^2(t)\rangle\rangle$ is small. The spikes of the second cumulant correspond to the short time intervals at which the distribution acquires a bimodal character while jumping from one well to the other. For the case of maximum amplification ($D=0.074$), the shape of the average is somewhat distorted with respect to the input and the spikes of the second moment are a little bit broader than for the $D=0.1$ case as can be seen in Fig. 4. On the other hand, as R is maximum for

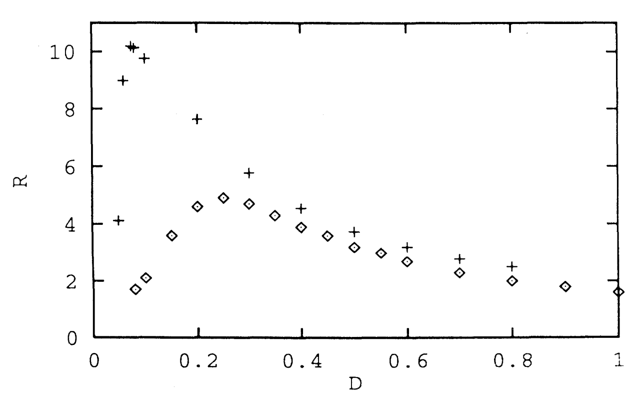

FIG. 2. Amplification factor R vs noise strength D for two values of the input fundamental frequency $\Omega=0.001$ (+ signs) and $\Omega=0.1$ (diamonds) for $S=0.1$.

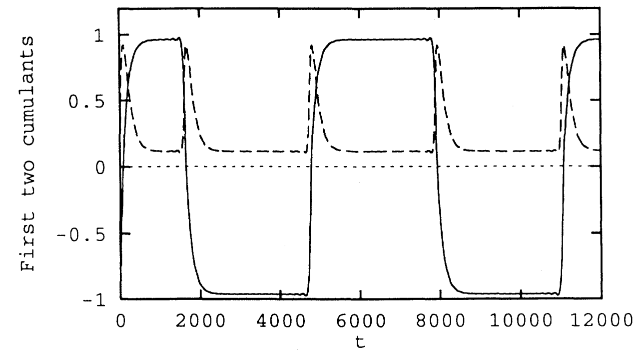

FIG. 3. Time evolution of the first two cumulants $\langle x(t) \rangle$, (solid line) and $\langle\langle x^2(t) \rangle\rangle$ (dashed lines) for $\Omega=0.001$, $S=0.1$, and $D=0.1$.

$D=0.074$, the distribution has to be narrower for this value than for $D=0.1$ so the minimum value for $\langle\langle x^2(t) \rangle\rangle$ in Fig. 4 is smaller than that in Fig. 3. For large values of D, the shape of the rectangular input is maintained but the degree of amplification decreases. The diffusive effect of the noise is very strong and it is felt all through the period of the external force. The distribution function shows two fairly broad maxima and so, the second cumulant is almost constant and rather large. On the other hand, for small noise strengths we have a strong distortion of the signal as shown in Fig. 5 for $D=0.05$. Here, Kramers rate is too small and, therefore, the degree of amplification should be smaller than in Figs. 3 and 4. The second moment is large as the distribution function is always double humped. Fourier analysis of the average indicates that new harmonics are not generated, so the distortion of the shape is due to the dispersive character of the response. Then we see that very low frequency input signals of a rectangular shape can be greatly amplified with very little distortion. This is not a universal feature in the sense that for a very low frequency sinusoidal signal, the degree of amplification is still very large but the shape of the output is no longer sinusoidal but almost rectangular.

Next, we present the results obtained with external field parameters such that LRT provides an adequate

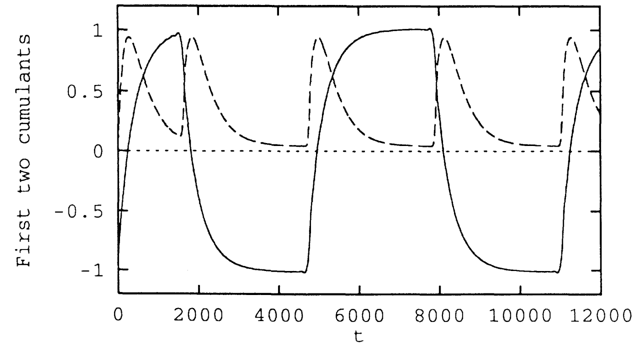

FIG. 4. Same as in Fig. 3 for $D=0.074$.

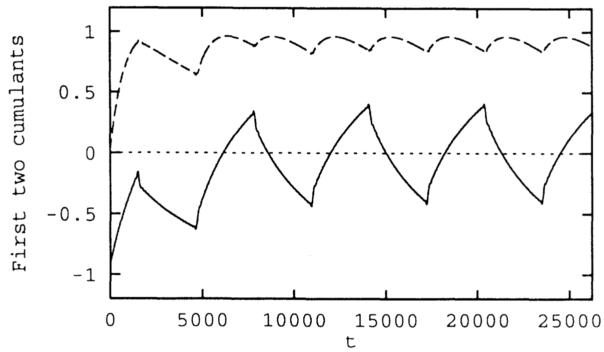

FIG. 5. Same as in Fig. 3 for $D=0.05$.

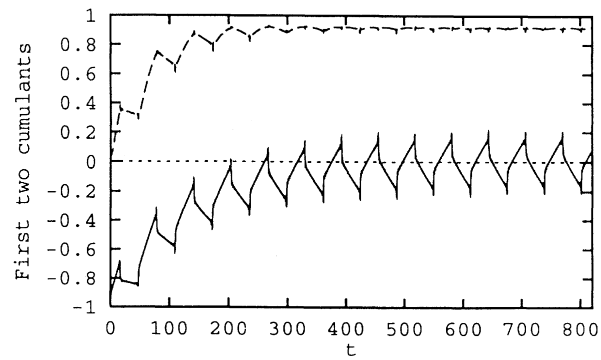

FIG. 6. Same as in Fig. 3 for $\Omega=0.1$ and $D=0.1$.

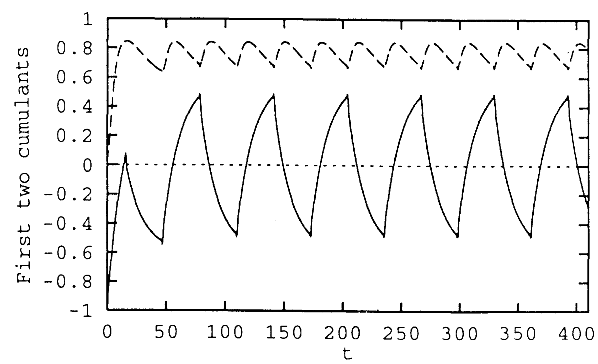

FIG. 7. Same as in Fig. 3 for $\Omega=0.1$ and $D=0.25$.

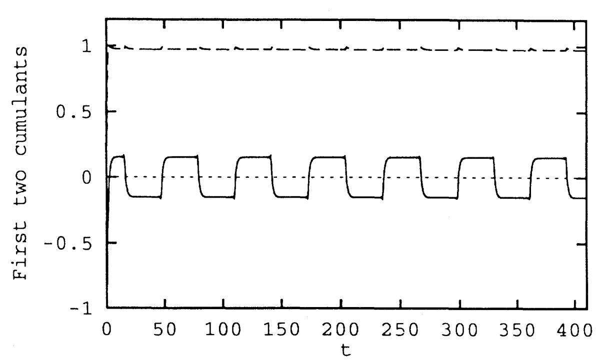

FIG. 8. Same as in Fig. 3 for $\Omega=0.1$ and $D=1.0$.

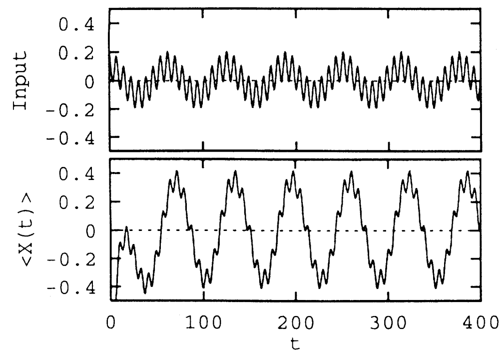

FIG. 9. Plots of the driving field $f(t)=0.1[\cos(0.1t)+\cos(0.8t)]$ and the average response for $D=0.3$.

description of the system dynamics as long as the noise values are not very small. By contrast with the situations considered before, the distribution function $P_\infty(x,t)$ will remain bimodal, being only slightly perturbed with respect to the equilibrium distribution in the absence of driving force. The dependence of the amplification factor R for a rectangular input with $S=0.1$ and $\Omega=0.1$ is shown in Fig. 2. The maximum amplification takes place at $D\approx0.25$ and, as it is clear from the plots, the degree of amplification is more modest than for the very low frequency case.

In Fig. 6, the time evolution of the first two cumulants is shown for $D=0.1$. There is a small amplification of the amplitude and quite a big distortion of the rectangular shape of the input. Fourier analysis of the output signal reveals that this distortion is due to the different degrees of amplification of the initial harmonics because of the dispersivity of the system and not to the generation of new ones. The second cumulant is quite large and almost constant. For maximum amplification, the distortion of the signal is still large as shown in Fig. 7. As in the previous case, the fundamental harmonics have the largest amplification, thus causing the change in the shape. Because of the resonant behavior of $\langle x(t)\rangle_\infty$, the second cumulant shows oscillations such that the fluctuations are minima when the average reaches its maximum amplitude in one period. Still, the probability density is bimodal and, therefore, the average cannot be amplified as much as in the very low frequency case where the population is concentrated alternatively in one of the two wells for most of the period. Finally, when D is large, the behavior of the system is qualitatively the same as for the very low frequency case. The amplification is small and the shape does not change much as shown in Fig. 8 for $D=1.0$.

The dispersive character of the response of the system for a fixed noise intensity leading to a selective amplification of the low frequency components of an input signal can be used to filter out the high frequencies. In Fig. 9, we show a driving field $f(t)=0.1[\cos(0.1t)+\cos(0.8t)]$ and the response of the system for $D=0.3$. Clearly, the high frequency component has a much weaker contribution to the output than the low frequency.

In conclusion, in this work we have analyzed the influence of noise and nonlinearity in the response of a system to a periodic driving field of rectangular shape. The analysis shows that for parameter values such that the LRT is valid, it is possible to amplify the signal, but because of the frequency dependence of the system susceptibility, this amplification is accompanied by a distortion of its shape. New harmonics are not generated by the system dynamics and so the response to a single frequency input would be an output with the same frequency but with its phase shifted with respect to that of the driver [3,4]. For very low frequency rectangular signals, a stochastic resonant behavior still exists. The weak driving field perturbs greatly the dynamics. The long time distribution function does not remain bimodal at all times for a range of values of the noise strength. Because of this, the noise power can be efficiently used to obtain a large amplification with a minimum distortion of the rectangular shape.

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